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A non-perturbative approach to the random-bond Ising model

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Abstract. We study the $N \to 0$ limit of the O(N) Gross–Neveu model in the framework of the massless form-factor approach. This model is related to the continuum limit of the Ising model with random bonds via the replica method. We discuss how this method may be useful in calculating correlation functions of physical operators. The identification of non-perturbative fixed points of the O(N) Gross–Neveu model is pursued by its mapping to a Wess–Zumino–Witten model.

1. Introduction

The aim of this paper is to discuss the critical regime of the two-dimensional random bond Ising model by applying non-perturbative methods to the massless phase of the O(N) Gross–Neveu (GN) model in the limit $N \to 0$. It is well known (and briefly sketched below) that in the context of the replica method the analytic continuation $N \to 0$ of the GN model describes the quenched averages of the Ising model in the presence of Gaussian distributed random couplings [1–5]. Perturbative calculations based on the GN model have then been extensively and successfully used for studying the behaviour of correlation functions of the random model in the infrared regime where the GN model for $N < 2$ is asymptotically free [2, 4, 5]. Being an asymptotically free infrared theory, the perturbative series is plagued by the presence of Landau poles which do not permit one to study the behaviour of the correlators on their short-distance scales and therefore to follow in general the change of the theory in passing from the short- to the large-distance scales. In this paper we show how in some cases we may get around this problem by using the non-perturbative massless form-factor (FF) approach proposed in [6]. For the two-point function of the energy operator of the random model we show for instance that the absence of Landau poles in the S-matrix formulation allows us to have much more information on the correlator at intermediate (and small) scales than what can be obtained within a perturbative renormalization group approach. A more general theoretical problem consists of identifying the ultraviolet fixed point of the GN model in its entire massless range $N < 2$. At the end of this paper we present some considerations on a mapping of the GN model to an interacting Wess–Zumino–Witten (WZW) model which may be useful for further investigation of this problem.

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2. S-matrix in replica space

Let us begin our discussion with the continuum limit of the random-bond Ising model expressed in terms of a Majorana fermion (see for example [1] for details) with the partition function given by

$$\mathcal{Z}[m(x)] = \int \mathrm{D}[\psi] \exp\left[- \int \mathrm{d}^2 x \, \overline{\psi} (\partial\!\!\!/ - m(x)) \psi \right]. \tag{2.1}$$

In the above formula, $m(x)$ is a (position-dependent) random mass term associated to the lattice random bond interactions. We assume that $m(x)$ has a Gaussian distributed probability $P[m(x)] \propto \exp(-m^2(x)/4g)$. Let us now use the standard replica method to compute the average of the free energy:

$$\overline{\ln \mathcal{Z}} = \lim_{N \to 0} \frac{\overline{\mathcal{Z}^N - 1}}{N}. \tag{2.2}$$

The quenched averages of correlation functions can then be obtained by adding a source term to the partition function (2.1) and differentiating (2.2) with respect to it. The above procedure leads to an effective action described by the GN model with $N \to 0$ colours:

$$\mathcal{S} = \int \mathrm{d}^2 x \, [\overline{\psi}_a \partial\!\!\!/ \psi_a + g(\overline{\psi}_a \psi_a)^2] \qquad a = 1, \ldots, N. \tag{2.3}$$

Irrespective of the number of colours N, the GN model presents at the classical level an infinite number of local and non-local charges which are conserved just in virtue of the equations of motion [7, 8]. For $N > 2$, the integrability at the classical level is known to survive at the quantum level: the associated quantum field theory is asymptotically free in the ultraviolet region, massive [9] and presents in general quite a rich spectrum of bound states. The integrability of the quantum model implies the elasticity and factorization of its S-matrix amplitudes which can then be determined by employing the unitarity and crossing symmetry equations, together with the residue equations coming from the bootstrap principle [10, 11].

For $N < 2$ the model is instead asymptotically free in the infrared region and presents at the quantum level a massless phase, as it can be argued (at least perturbatively) from the changing of the sign of the β-function near $g = 0$ [12]. Its spectrum consists in this case of only N massless Majorana fermions which can be right- or left-movers with a dispersion relation parametrized in terms of a crossover scale M and the rapidity variable as $E = p = (M/2)\exp(\theta)$ and $E = -p = (M/2)\exp(-\theta)$ respectively [13]. Assuming that the integrability† of the model also holds for $N < 2$, we propose the following exact S-matrix involving its fermionic massless excitations (for the general formulation of massless scattering theories, see [15])‡

$$S_{R,R} = S_{L,L} = S_{R,L} = S \tag{2.4}$$

where

$$S_{a,b}^{c,d}(\theta) = \delta_{a,b}\delta^{c,d}\sigma_1(\theta) + \delta_a^c \delta_b^d \sigma_2(\theta) + \delta_a^d \delta_b^c \sigma_3(\theta) \tag{2.5}$$

† This statement can actually be proven along the lines of [14] if one is inclined to be rather cavalier about the meaning of the colour indices in this range of N.

‡ This essentially follows the original proposal made in [13] for the massless S-matrix of the GN model, with the difference that there the S-matrix in the L–L and R–R channel was incorrectly assumed to be -1 on the basis of Feynman rules derived from the Lagrangian (2.3) which, however, involves for $N < 2$ irrelevant non-renormalizable interactions.

with

$$\sigma_1(\theta) = -\frac{\mathrm{i}\lambda}{\mathrm{i}\pi - \theta}\sigma_2(\theta) \qquad \sigma_3(\theta) = -\frac{\mathrm{i}\lambda}{\theta}\sigma_2(\theta)$$
$$\sigma_2(\theta) = -\frac{\Gamma\left(1-\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2\pi\mathrm{i}}-\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(-\frac{\lambda}{2\pi\mathrm{i}}+\frac{\theta}{2\pi\mathrm{i}}\right)}{\Gamma\left(\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2\pi\mathrm{i}}+\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}-\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(1-\frac{\lambda}{2\pi\mathrm{i}}-\frac{\theta}{2\pi\mathrm{i}}\right)} \tag{2.6}$$

and

$$\lambda = \frac{2\pi}{N-2}. \tag{2.7}$$

The above expressions formally coincide with those discussed in [11] for the massive phase of the model, the main distinctions being the different role played by the rapidity parameter as well as the different range of the parameter N in the two cases. The ansatz (2.6) is supported by the validity of the Yang–Baxter equations for all possible combinations of right- and left-moving particles as well as by the $\mathrm{O}(N)$-invariance of the interaction. Notice that for $N < 2$ the above amplitudes do not have poles in the physical sheet (a feature which is consistent with the massless phase of the model). For $N \to 2$ they are continuous functions and at $N = 2$ reduce to the S-matrix amplitudes of the Sine-Gordon model at the Coleman transition point $\beta^2 \to 8\pi$

$$\begin{aligned}\sigma_2(\theta) &= 0\\ \sigma_1(\theta) &= \frac{-2\mathrm{i}\pi}{\mathrm{i}\pi-\theta}\,\frac{\Gamma\left(1-\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi\mathrm{i}}\right)}{\Gamma\left(\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}-\frac{\theta}{2\pi\mathrm{i}}\right)}\\ \sigma_3(\theta) &= \frac{-2\mathrm{i}\pi}{\theta}\,\frac{\Gamma\left(1-\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi\mathrm{i}}\right)}{\Gamma\left(\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}-\frac{\theta}{2\pi\mathrm{i}}\right)}.\end{aligned} \tag{2.8}$$

3. Minimal FFs

Let us now consider the calculation of correlation functions by means of the spectral representation based on the FFs (see for instance [6, 16, 17] for the relevant formulae and notation). For the two-particle FF the problem can be initially reduced to consider the three different channels whose S-matrices are given by:

$$\sigma_{\mathrm{iso}} = N\sigma_1 + \sigma_2 + \sigma_3 \qquad \sigma_\pm = \sigma_2 \pm \sigma_3. \tag{3.1}$$

For each of these channels, the minimal FF can be evaluated. As an example we construct the one for the isoscalar channel. Watson's equations for a massless flow are [6] (see also [18]):

$$F_{\alpha_1,\alpha_2}(\theta) = S_{\alpha_1,\alpha_2}(\theta)F_{\alpha_2,\alpha_1}(-\theta) \qquad F_{\alpha_1,\alpha_2}(\theta+2\pi\mathrm{i}) = F_{\alpha_2,\alpha_1}(-\theta) \tag{3.2}$$

where $\alpha_i = R, L$ and S stands for a scalar S-matrix which in our case is σ_{iso}. In order to solve these equations it may be useful to employ an integral representation for σ_{iso}:

$$\sigma_{\mathrm{iso}}(\theta) = -\exp\left\{-2\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\mathrm{e}^{(\frac{\lambda}{\pi}+1)x}-\mathrm{e}^{-x}}{\mathrm{e}^x+1}\sinh\left(\frac{\theta x}{\mathrm{i}\pi}\right)\right\}. \tag{3.3}$$

(Note that for $N = 1$ this simply reduces to -1—the S-matrix of the Ising model.) Let us write $F_{\alpha_1,\alpha_2}(\theta)$ as

$$F_{\alpha_1,\alpha_2}(\theta) = f_{\alpha_1,\alpha_2}(\theta)g_{\alpha_1,\alpha_2}(\theta) \tag{3.4}$$

where $g_{\alpha_1,\alpha_2}(\theta)$ satisfies

$$g_{\alpha_1,\alpha_2}(\theta) = -g_{\alpha_2,\alpha_1}(-\theta) \qquad g_{\alpha_1,\alpha_2}(\theta + 2\pi\mathrm{i}) = g_{\alpha_2,\alpha_1}(-\theta) \tag{3.5}$$

whereas $f_{\alpha_1,\alpha_2}(\theta)$ fulfils the equations

$$\begin{aligned} f_{\alpha_1,\alpha_2}(\theta) &= \exp\left\{-2\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\mathrm{e}^{(\frac{\lambda}{\pi}+1)x} - \mathrm{e}^{-x}}{\mathrm{e}^x + 1}\sinh\left(\frac{\theta x}{\mathrm{i}\pi}\right)\right\} f_{\alpha_2,\alpha_1}(-\theta) \\ f_{\alpha_1,\alpha_2}(\theta + 2\pi\mathrm{i}) &= f_{\alpha_2,\alpha_1}(-\theta). \end{aligned} \tag{3.6}$$

The solution of (3.5) for the R–R and L–L channels is unique (up to a normalization constant) and is given by

$$g_{R,R}(\theta) = g_{L,L}(\theta) = \sinh\left(\frac{\theta}{2}\right). \tag{3.7}$$

For the R–L channel any linear combination of $\mathrm{e}^{\frac{\theta}{2}}$ and $\mathrm{e}^{-\frac{\theta}{2}}$ is a solution. The exact computation of the two-particle FF for the energy operator for $N = 1$ (which corresponds to the pure Ising model) and its expected infrared behaviour at $N = 0$ fixes the solution in a unique way

$$g_{R,L}(\theta) = \mathrm{e}^{\frac{\theta}{2}}. \tag{3.8}$$

As for (3.6) we take the same solution in the R–R, L–L and R–L channels (this assumes that $f_{R,L}(\theta) = f_{L,R}(\theta)$):

$$f_{R,R} = f_{L,L} = f_{R,L} = f \tag{3.9}$$

with the function f given by

$$f(\theta) = \exp\left\{-2\int_0^\infty \frac{\mathrm{d}x}{x}\,\frac{\mathrm{e}^{(\frac{\lambda}{\pi}+1)x} - \mathrm{e}^{-x}}{\mathrm{e}^x + 1}\,\frac{\sin^2\left(\frac{x(\mathrm{i}\pi-\theta)}{2\pi}\right)}{\sinh x}\right\}. \tag{3.10}$$

This can also be expressed as

$$\begin{aligned} f(\theta) = &\prod_{k=0}^{\infty}\left(\frac{\Gamma\left(1-\frac{\lambda}{2\pi}+k\right)\Gamma\left(\frac{3}{2}+k\right)}{\Gamma(2+k)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2\pi}+k\right)}\right)^2 \\ &\times\frac{\Gamma\left(\frac{5}{2}-\frac{\theta}{2\pi\mathrm{i}}+k\right)\Gamma\left(\frac{3}{2}+\frac{\theta}{2\pi\mathrm{i}}+k\right)\Gamma\left(1-\frac{\lambda}{2\pi}-\frac{\theta}{2\pi\mathrm{i}}+k\right)\Gamma\left(-\frac{\lambda}{2\pi}+\frac{\theta}{2\pi\mathrm{i}}+k\right)}{\Gamma\left(2-\frac{\theta}{2\pi\mathrm{i}}+k\right)\Gamma\left(1+\frac{\theta}{2\pi\mathrm{i}}+k\right)\Gamma\left(\frac{3}{2}-\frac{\lambda}{2\pi}-\frac{\theta}{2\pi\mathrm{i}}+k\right)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2\pi}+\frac{\theta}{2\pi\mathrm{i}}+k\right)}. \end{aligned} \tag{3.11}$$

Using its infinite-product representation one can show that $f(\theta)$ satisfies the following functional relation:

$$f(\theta)f(\theta+\mathrm{i}\pi) = \mathcal{C}_\lambda\frac{\Gamma\left(-\frac{\lambda}{2\pi}+\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2\pi}-\frac{\theta}{2\pi\mathrm{i}}\right)}{\Gamma\left(1+\frac{\theta}{2\pi\mathrm{i}}\right)\Gamma\left(\frac{3}{2}-\frac{\theta}{2\pi\mathrm{i}}\right)} \tag{3.12}$$

which may be useful in the computation of higher-particle FFs. The constant $\mathcal{C}_\lambda$ is given by

$$\mathcal{C}_\lambda = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(-\frac{\lambda}{2\pi}\right)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2\pi}\right)}\prod_{k=0}^{\infty}\left(\frac{\Gamma(1-\frac{\lambda}{2\pi}+k)\Gamma\left(\frac{3}{2}+k\right)}{\Gamma(2+k)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2\pi}+k\right)}\right)^4\frac{(1+k)\left(\frac{3}{2}+k\right)}{\left(-\frac{\lambda}{2\pi}+k\right)\left(\frac{1}{2}-\frac{\lambda}{2\pi}+k\right)} \tag{3.13}$$

its numerical value at $N = 0$ is $\mathcal{C}_{-\pi} = 0.5854\ldots$.

4. Two-point function of the energy operator

Let us specialize our analysis to evaluate the (average) correlation function of the energy operator $\overline{\langle\epsilon(x)\epsilon(0)\rangle}$. In terms of the replica, this is equivalent to evaluating the correlation function

$$\frac{1}{N}\sum_{k=1}^{N}\langle\epsilon_k(x)\epsilon_k(0)\rangle|_{N=0} \tag{4.1}$$

in the GN model. At the infrared fixed point, the energy operator is given in terms of the fermions by $\epsilon_k(x) \sim -\mathrm{i}\psi_k^{(-)}(x)\psi_k^{(+)}(x)$ where $\psi_k^{(-)}(x)$ and $\psi_k^{(+)}(x)$ are the chiral components of the original Majorana fermions. Duality and spin reversal symmetry translate into invariance of the Lagrangian (2.3) under the two transformations $\psi_k^{(+)} \mapsto -\psi_k^{(+)}$ and $\psi_k^{(-)} \mapsto \psi_k^{(-)}$ or $\psi_k^{(+)} \mapsto \psi_k^{(+)}$ and $\psi_k^{(-)} \mapsto -\psi_k^{(-)}$. Under these two transformations, the energy operator changes its sign. Since these discrete symmetries are preserved by the perturbation, one concludes that the only non-vanishing FFs of the energy-operator are those with an odd number of both left- and right-moving particles. In particular, the first non-trivial FF is

$$F_{i,j}^{R,L}(\theta_1,\theta_2) = \langle 0|\left(\sum_{k=1}^{N}\epsilon_k(0)\right)|R_i(\theta_1)L_j(\theta_2)\rangle. \tag{4.2}$$

Here $|R_i(\theta_1)L_j(\theta_2)\rangle$ corresponds to an asymptotic state of one right-moving and one left-moving fermionic particle. $\mathrm{O}(N)$ and Lorentz invariance fixes this FF (up to an overall normalization constant) to be given by $F_{i,j}^{R,L}(\theta_1,\theta_2) = \delta_{i,j}F_{R,L}(\theta_1-\theta_2)$ with $F_{R,L}$ as in equations (3.4), (3.8) and (3.10). In the previous expression there could be, in principle, a supplementary factor (a $2\pi\mathrm{i}$ periodic symmetric function without poles, i.e. a polynomial in e^{θ_i}). This factor is actually absent in the pure Ising model ($N = 1$) and its absence will be further justified later by the correct asymptotic behaviour of the correlation function (4.1). The contribution of (4.2) to the two-point correlation function of the energy operator is given by

$$\begin{aligned}\langle\epsilon(x)\epsilon(0)\rangle^{(2)} &\sim \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta_2}{2\pi}|F_{R,L}(\theta_1-\theta_2)|^2\exp\left(-\frac{Mr}{2}[\mathrm{e}^{\theta_1}+\mathrm{e}^{-\theta_2}]\right)\\ &\equiv C_\epsilon^{(2)}(Mr)\end{aligned} \tag{4.3}$$

where the superscript indicates that this is the two-particle contribution and $x = (\mathrm{i}r, 0)$†. With the change of variables $\gamma = \theta_1+\theta_2$, $\theta = \theta_1-\theta_2$ we obtain

$$\begin{aligned}C_\epsilon^{(2)}(Mr) &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\gamma}{2\pi}|F_{R,L}(\theta)|^2\exp\left(-Mr\mathrm{e}^{\frac{\theta}{2}}\cosh\frac{\gamma}{2}\right)\\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{2\pi^2}K_0(Mr\mathrm{e}^{\frac{\theta}{2}})|F_{R,L}(\theta)|^2.\end{aligned} \tag{4.4}$$

Using (3.4), (3.8)–(3.10) we can express $|F_{R,L}(\theta)|^2$ as (hereafter we will specialize our formulae to the limit $N \to 0$)

$$|F_{R,L}(\theta)|^2 = \frac{\mathrm{e}^{\theta}}{\sqrt{1+\frac{\theta^2}{\pi^2}}}\Phi(\theta) \tag{4.5}$$

† Actually, there is another two-particle contribution to this correlation function, namely the one coming from $F_{L,R}$. However, it gives the same contribution as $F_{R,L}$.

with

$$\Phi(\theta) = \exp\left\{2\int_0^\infty \frac{dx}{x} e^{-x} \tanh^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x\theta}{2\pi}\right)\right\}. \tag{4.6}$$

It is straightforward to evaluate this last integral numerically, it is bounded from below by 0 and from above by

$$2\int_0^\infty \frac{dx}{x} e^{-x}\tanh^2\left(\frac{x}{2}\right) = 0.315\,384\ldots \tag{4.7}$$

i.e.

$$1 \leqslant \Phi(\theta) \leqslant 1.370\,786 \tag{4.8}$$

for any value of θ. In particular, the factor $\Phi(\theta)$ will not influence the $\theta \to \pm\infty$ asymptotics of the FF. Thus, we can replace $\Phi(\theta)$ by a constant in order to discuss the infrared behaviour of the contribution (4.4) to the correlation function. Putting

$$\Phi(\theta) = 1 \tag{4.9}$$

we find

$$C_\epsilon^{(2)}(Mr) = \int_0^\infty dp/\pi^2 \frac{p}{\sqrt{1+4\ln^2 p/\pi^2}} K_0(Mrp) \qquad Mr \gg 1 \tag{4.10}$$

where we have made the change of variables $p = e^{\frac{\theta}{2}}$. Upon expansion of the integrand, the leading infrared behaviour is found to be given by

$$C_\epsilon^{(2)}(Mr) = \frac{1}{2\pi(Mr)^2 \ln Mr}\left(1 + O\left(\frac{1}{\ln Mr}\right)\right). \tag{4.11}$$

This fits nicely with renormalization group results, as the following discussion shows. Up to three-loop order, we have (see e.g. the two-loop result equation (4.10) of [5] which we have extended by one further order using results of [19]):

$$\begin{aligned}\langle\epsilon(x)\epsilon(0)\rangle = r^{-2}\frac{4g_0}{\mathcal{N}\zeta}\Bigg\{1 &+ \frac{2\pi g_0(1+2\ln(\zeta)-\zeta)}{\zeta} \\ &+ \frac{2\pi^2 g_0^2\left(3\zeta^2 - 8\zeta + 5 + 8\ln(\zeta)^2 - 4\ln(\zeta)\zeta\right)}{\zeta^2}\Bigg\}\end{aligned} \tag{4.12}$$

where $\zeta = 1 + 8\pi g_0 \ln r$ and g_0 is a bare coupling constant. The normalization constant $\mathcal{N}$ in (4.12) should be chosen such that this perturbative result agrees with (4.11) at $M = 1$ for $r \to \infty$. Truncation of (4.12) at one, three or two loops leads to $\mathcal{N} = 1$, $1 - 2\pi g_0$ and $1 - 2\pi g_0 + 6\pi^2 g_0^2$, respectively. Setting $M = 1$ we obtain a good agreement with (4.4) in the region $Mr \geqslant 10$ with $g_0 = 0.23$, 0.038 and 0.050 for the truncation of (4.12) at order g_0, g_0^2 and g_0^3, respectively. One can see that the higher orders shift the Landau pole towards smaller scales and make the behaviour of (4.12) more regular around $Mr = 1$ in figure 1. Even higher loop corrections in the pertubative framework are expected to move this pole to smaller and smaller scales. The main point is the absence of this kind of singularity in our result based on the FF approach.

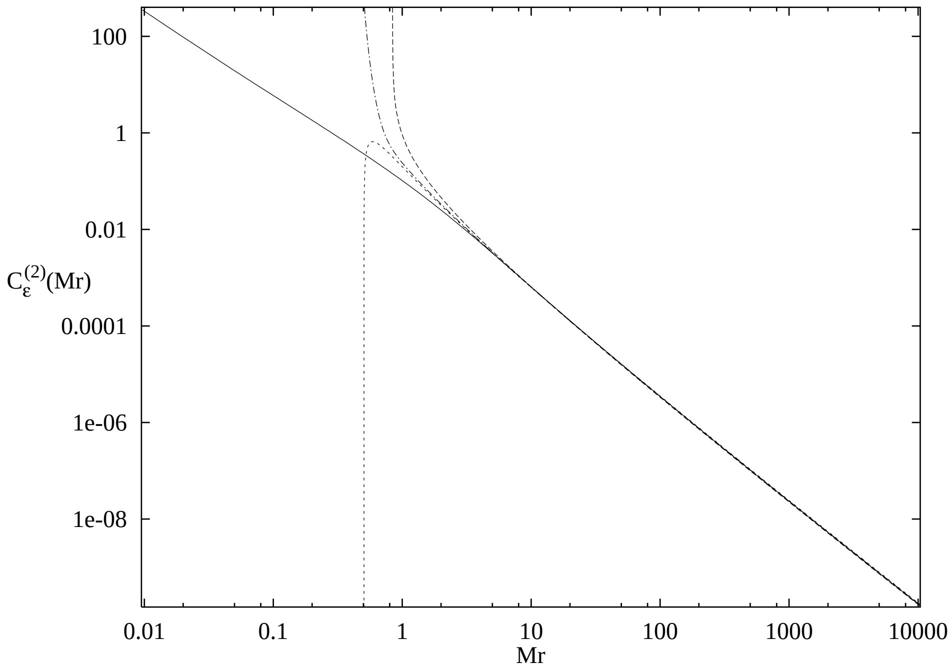

Figure 1. Two-particle approximation to the correlation function of the energy operator (full line), the one-loop perturbative result (long broken curve), the two-loop result (short broken curve) and the three-loop result (chain curve).

5. Comments on higher-particle FFs

Apart from logarithmic corrections, the energy operator has canonical scaling dimensions in the infrared and, as explicitly checked for similar operators in other massless scattering theories, a rapidly convergent spectral series for its correlation functions is expected in this case [6, 20]. This expectation seems indeed confirmed by the above calculation. Under these favourable circumstances, one is essentially released from computing the more complicated and cumbersome expressions of the higher-particle FFs for most reasonable purposes. It should be pointed out though, that these favourable cases do not always occur. Consider, for instance the (averaged) two-point correlation function of the spin operator. In this case two kinds of technical problems arise: the first difficulty is that the integrals entering the spectral series should be regularized in order to cure the infrared divergencies. This is, however, a well-known problem of the spectral representation based on massless FFs [6], a problem that now can be easily handled following the suggestion of [21]. The second difficulty is the most serious one, namely the apparent necessity of employing in this case all higher-particle FFs of the spin operator in order to recover both its infrared and ultraviolet behaviour. The problem is particularly difficult here since there is a huge difference in computing higher-particle FFs in diagonal rather than non-diagonal scattering theories (in the diagonal case one is often able to obtain a closed expression for all the higher-particle FFs, see for instance [22]). Although progress has recently been achieved on the FF problem in some non-diagonal scattering theories [18, 23], the general determination of the higher-particle FFs for non-diagonal S-matrix models still remains an open problem of a (quite) mathematical difficulty. It would be clearly interesting to try to develop the massless FF approach further in order to deal successfully with this class of operators.

6. Non-trivial fixed points

In the last part of this paper, we present some considerations which may be useful to better understand the structure of the fixed points in the O(N) GN model and in particular to determine its ultraviolet fixed point in the range $N < 2$. The theory behaves once again differently for $N > 2$ and for $N < 2$. More is known about the structure of the fixed points of the former case rather than of the latter. Let us actually recall that for $N > 2$ the model is massive and ultraviolet asymptotically free. This means that at short distances it reduces to N free Majorana fields (with central charge $c_{\mathrm{uv}} = N/2$) whereas at large distance scales no massless degrees of freedom are left and its central charge is therefore $c_{\mathrm{ir}} = 0$. Conversely, for $N < 2$ the model is asymptotically free in its infrared scales and there we have correspondingly $c_{\mathrm{ir}} = N/2$. However, since in this case the model is massless along all its flow from large to short distance scales, the central charge in the ultraviolet limit is one of the dynamical data which remain to be determined. Some insight can be gained by rewriting the partition function of the model defined by the action (2.3) in a way more appropriate for studying the existence of some other fixed point in addition to the (trivial) one at $g = 0$ (a similar procedure has been used in [24] to study non-trivial fixed points in the chiral $SU(N)$ GN model). To this end, we first use the identity

$$(\bar{\psi}_a\psi_a)^2 = -2(\bar{\psi}_a\gamma_\mu T^A_{ab}\psi_b)^2 \tag{6.1}$$

where T^A are the O(N) generators normalized as $\mathrm{tr}T^AT^B = \delta^{AB}$ (this identity is a direct consequence of the fact that we are working with Majorana fermions). Then the quartic term in the interaction can be traded for a quadratic term via the introduction of an auxiliary field through the identity

$$\exp\left(2g\int \mathrm{d}^2x\,(\bar{\psi}_a\gamma_\mu T^A_{ab}\psi_b)^2\right) = \int \mathrm{D}A^A_\mu \exp\left(\frac{1}{8g}\int \mathrm{d}^2x\,(A^A_\mu)^2 + \mathrm{i}\int \mathrm{d}^2x\,(\bar{\psi}\gamma_\mu T^A\psi)A^A_\mu\right) \tag{6.2}$$

and hence the partition function can be written as

$$\mathcal{Z} = \int \mathrm{D}\bar{\psi}_a\,\mathrm{D}\psi_a\,\mathrm{D}A^A_\mu\,\exp\left[-\int \mathrm{d}^2x\left(\bar{\psi}_a(\partial\!\!\!/\delta_{ab} + \mathrm{i}A\!\!\!/_{ab})\psi_b + \frac{1}{8g}A^A_\mu A^A_\mu\right)\right]. \tag{6.3}$$

The key observation now is that the fields in equation (6.3) can be decoupled through the transformations

$$A = -\partial hh^T \qquad \bar{A} = -\bar{\partial}gg^T \qquad \psi = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}\chi \tag{6.4}$$

where h and g are O(N) matrix-valued fields, and ^{T} stands for transpose. Taking into account the Jacobians of the above transformations [25] (for more details compare also with section 9 of [26]), we can rewrite the whole partition function as a product of decoupled sectors

$$\mathcal{Z} = \mathcal{Z}_{\mathrm{ff}}\mathcal{Z}_{\mathrm{gh}}\mathcal{Z}_{\mathrm{int}} \tag{6.5}$$

where $\mathcal{Z}_{\mathrm{ff}}$ is the partition function for the N free Majorana fermions, $\mathcal{Z}_{\mathrm{gh}}$ the ghost partition function arising in the computation of the Jacobians and

$$\mathcal{Z}_{\mathrm{int}} = \int \mathrm{D}h\,\mathrm{D}g\,\exp\Bigg\{(1+2C_v)(\Gamma[h] + \Gamma[g^T]) - \left(\frac{(1+2C_v)\alpha}{2\pi} + \frac{1}{8g}\right)\int \mathrm{d}^2x\,\mathrm{tr}(\partial hh^T\bar{\partial}gg^T)\Bigg\} \tag{6.6}$$

where α is a parameter that keeps track of regularization ambiguities, C_v is the dual Coxeter number of $O(N)$ and $\Gamma[u]$ is the WZW action [27]. Hence, modulo decoupled conformally invariant sectors, we have an effective theory of interacting WZW fields.

One comment is in order regarding the regularization ambiguities arising in the evaluation of the Jacobians. The quadratic term in A_μ^A in (6.3) breaks both gauge invariance and local chiral invariance, hence there is no *a priori* reason to choose a regularization preserving any of these two symmetry transformations. We will fix the parameter α later. Using the Polyakov–Wiegmann identity [28] it is straightforward to see the existence of a fixed point given by the value of the coupling constant $g_1 = \pi/(4(1+2C_v)(1-\alpha))$. The effective partition function at this value may be written as

$$\mathcal{Z}_{\mathrm{int}}|_{g_1} = \int \mathrm{D}\tilde{g}\ \exp((1+2C_v)\Gamma[\tilde{g}]) \tag{6.7}$$

where the identification $\tilde{g} = g^T h$ has been made and the integral over h has been factored out. Taking into account the free fermions and the ghosts, the partition function (6.5) corresponds to a conformal field theory whose Virasoro central charge is given by

$$c_1 = \frac{N}{2} - N(N-1) + \left[\frac{(1+2C_v)N(N-1)}{2(1+C_v)}\right] = \frac{N}{2}\left(1 - \frac{N-1}{1+C_v}\right). \tag{6.8}$$

There may be, however, another fixed point, given by the value $g_2 = -\pi/(4(1+2C_v)\alpha)$, where we have

$$\mathcal{Z}_{\mathrm{int}}|_{g_2} = \int \mathrm{D}h\,\mathrm{D}g\ \exp\{(1+2C_v)(\Gamma[g^T] + \Gamma[h])\} \tag{6.9}$$

and the corresponding Virasoro central charge is given by

$$c_2 = \frac{N}{2} - N(N-1) + 2\left[\frac{(1+2C_v)N(N-1)}{2(1+C_v)}\right] = \frac{N}{2} + N(N-1)\frac{C_v}{1+C_v}. \tag{6.10}$$

First we have to choose the parameter α and secondly identify the fixed points in the two ranges $N > 2$ and $N < 2$.

Let us first see how the above points can be settled in the well-understood case $N > 2$. Since in this region the model is asymptotically free in the ultraviolet and massive otherwise, no non-trivial fixed point is expected. Therefore one should choose $g = \infty$ in (6.3) since then the integration over the gauge fields leads to constraints that eliminates all the degrees of freedom yielding a $c = 0$ theory. This suggests the choice $\alpha = 1$ (i.e. the gauge-invariant regularization) for which we have an infrared fixed point in the strong coupling limit $g_1 \to \infty$. Plugging $C_v = N - 2$ into the corresponding expression (6.8) of the central charge, we find indeed the correct value $c_1 = 0$. Notice that with this choice of α, the value of g_2 is negative and does not correspond to a physical fixed point, because this is incompatible with the unitarity of the GN model for $N > 2$.

The GN model at $N = 2$ is somewhat special since it is equivalent to the massless conformally invariant Thirring model with $c = 1$ for $-\frac{4}{\pi} \leqslant g < \infty$. At $N = 2$ one has $C_v = 0$ and retaining $\alpha = 1$ as before, one finds that $g_1 = \infty$ still corresponds to a theory with $c_1 = 0$ which is consistent with approaching $N = 2$ from above out of a massive regime. The other fixed point is now located at the end of the line where $c = 1$, i.e. $g_2 = -\frac{4}{\pi}$ and $c_2 = 1$ which seems to be more appropriate for the approach to $N = 2$ from below out of the massless regime.

Let us now discuss the fixed points in the regime $N < 2$ assuming that both formulae $\alpha = 1$ and $C_v = N - 2$ also apply here. The GN model is generally non-unitary in this regime, a fact which opens the possibility of also considering a fixed point with a negative value of the coupling constant. With the above choice of α and C_v, the central charge for the fixed point g_1 is identically zero also for $N < 2$. Let us consider the second critical value of the coupling constant, i.e. g_2. Note that g_2 is negative for $N > \frac{3}{2}$ but positive otherwise and therefore also compatible with the statistical interpretation as the random-bond Ising model for $N \to 0$. The corresponding value of the ultraviolet central charge is then given by (6.10), i.e. upon inserting $C_v = N - 2$

$$c_{\mathrm{uv}} = \frac{N(2N-3)}{2}. \tag{6.11}$$

The above formula does not seem to apply to the case $N = 1$: in fact it predicts $c_{\mathrm{uv}} = -\frac{1}{2}$, but instead in this case we expect to find $c_{\mathrm{uv}} = c_{\mathrm{ir}} = \frac{1}{2}$ for the simple reason that it is impossible to construct a quartic fermionic interaction with only one Majorana fermion. The reason of this mismatch seems somehow interesting: first notice that using (6.10) the variation of the central charge from the short to the large distances reads

$$\Delta c = N(N-1)\frac{C_v}{1+C_v}. \tag{6.12}$$

It is a general result of two-dimensional quantum field theories that such variations of central charges satisfy the sum rule [29]

$$\Delta c = \frac{3}{4\pi}\int \mathrm{d}^2x\, |x|^2 \langle\Theta(x)\Theta(0)\rangle \tag{6.13}$$

where $\Theta(x)$ is the trace of the stress-energy tensor. This operator is proportional to the quartic interaction term in the Lagrangian (2.3) (the proportionality constant being the $\beta(g)$ function of the model). Therefore, the term $N(N-1)$ in (6.12) has a pure combinatorial origin. This factor itself of course vanishes for $N = 1$. However, after pulling out this combinatorial term, what is left in (6.12) or in (6.13) may be interpreted as 'Δc per unit of replica', i.e. a quantity which still depends on N but which has lost any reference to the colour indices of the theory. Let us denote it by $D(N)$. For $N = 1$ this quantity can be easily computed by means of the Green function of the free fermion resulting in a (logarithmically) divergent expression

$$D(1) \sim \int \mathrm{d}^2x\, |x|^2 \frac{1}{|\,x\,|^4}. \tag{6.14}$$

For $N \neq 1$ we expect that the divergence is cured by the presence of interactions so that we expect $D(N)$ to be a finite quantity†. The above considerations suggest then for $N = 1$ there may be a sort of anomaly which conspires to give a non-zero value for Δc. In this case the correct procedure might be to set $N = 1$ in (6.10) before inserting the expression of C_v, i.e. to use a regularization expression for $D(N = 1)$.

7. Conclusions

In summary, in this paper we have applied massless FFs to the critical regime of the random-bond Ising model. Compared with the perturbative approach, the energy correlation function obtained with this method is well behaved in the whole range of scales. The technical difficulties encountered in the computation of higher-particle FFs can possibly be circumvented by using methods similar to those used in [18, 23], where FFs for some non-diagonal S-matrices have been computed. One motivation of our approach is to open the way for a more deep understanding of the ultraviolet behaviour of the massless flow of the

† At $N = 2$ it is the vanishing of the β function which is responsible for the vanishing of $D(2)$.

Ising model from its pure to its disordered regime and, more generally, for all the $O(N)$ GN models with $N < 2$ which appear in the replica approach of the original random problem. To this end we have also pointed out a possible way of reaching the ultraviolet fixed point by a mapping of the GN action to a WZW model, relating the strong and weak coupling regimes. This approach predicts the presence of a non-trivial fixed point with central charge given for $N \neq 1$ by (6.11). The lacking of a sound mathematical definition for the relevant quantities of the group $O(N)$ for $N < 2$ makes it highly interesting to have independent information on this issue. In this respect, the most natural approaches are those based on the thermodynamical Bethe ansatz [30] or the aforementioned c-theorem sum rule [29]. The application of the two approaches seems, however, presently obstructed by technical difficulties related both to the non-diagonal nature of the S-matrix and to the subtle problem of how to take the analytic continuation of the mathematical expressions for continuous values of N in the range $N < 2$ (this is particularly severe for a thermodynamical Bethe ansatz approach). The solution of these problems together with the massless FF approach proposed here may give interesting non-perturbative information in the field of disordered systems.

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